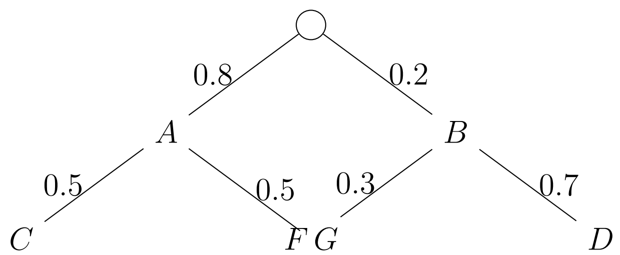
In this worksheet you will learn to use tree diagrams as a visual tool to represent sequences of events and calculate their probabilities.

Easy Questions

1. Consider a single coin toss. Write down the two possible outcomes and, on pen and paper, draw a simple probability tree diagram for this event. Then state the probability for each outcome assuming a fair coin, i.e. $\frac{1}{2}$ for each branch.
2. Consider two successive tosses of a fair coin. On pen and paper, draw a probability tree diagram. Using your diagram, compute the probability of obtaining two heads.
3. For two coin tosses, list all possible outcomes by drawing the corresponding tree diagram on pen and paper. Write down the probability associated with each outcome.
4. Using the probability tree diagram you created for two coin tosses, calculate the probability of obtaining exactly one head and one tail.
5. In a probability tree for a single event with two outcomes, if one branch has probability 0.7, what is the probability of the other branch? Explain your reasoning.

Intermediate Questions

6. A fair coin is tossed twice. On pen and paper, draw the probability tree diagram and calculate the probability of obtaining exactly one head. (Hint: count the branches leading to one head.)
7. Consider the following probability tree diagram:



   Calculate the probability of event C occurring.

8. Using the tree diagram from the previous question, calculate:

(a) the probability of event D occurring (regardless of branch), and

(b) the probability of following the path A then D.

9. On pen and paper, draw a probability tree diagram for two independent events where event X has outcomes: success with probability 0.6 and failure with probability 0.4, and event Y has outcomes: success with probability 0.7 and failure with probability 0.3. Compute the probability of obtaining two successes.

10. A machine performs two operations. The probability that operation 1 is successful is 0.9. If operation 1 is successful, the probability that operation 2 is successful is 0.8; if operation 1 fails, the probability that operation 2 is successful is 0.4. On pen and paper, draw a probability tree diagram and compute the overall probability that operation 2 is successful.

11. A probability tree is partially completed as follows. The first branch splits into events E and F, but the probability of E is unknown. The branch from E splits into G with probability 0.3 and H with probability 0.7. The branch from F splits into G with probability 0.5 and H with probability 0.5. If the overall probability of G is 0.4, determine the probability of branch E.

12. Consider three successive tosses of a fair coin. Using a pen and paper tree diagram, calculate the probability of obtaining exactly two heads.

13. A spinner is used in the following experiment. In the first spin, the spinner lands on outcome X with probability 0.6 and on outcome Y with probability 0.4. If X occurs, the second spin yields outcome Z with probability 0.5 and outcome W with probability 0.5. If Y occurs, the probabilities are 0.2 for Z and 0.8 for W. Compute the probability of obtaining outcome W after the two spins.

14. In a probability tree for a single event with two outcomes, if one branch is known to have probability 0.65, determine the probability of the missing branch. Explain why your answer is valid.

15. In a two-event probability tree, the first event has outcomes with probabilities $P(A) = 0.7$ and $P(B) = 0.3$. If A occurs then $P(C \mid A) = 0.6$, and if B occurs then $P(C \mid B) = 0.5$. Compute the overall probability of event C.

16. A bag contains 3 blue balls and 2 red balls. Two balls are drawn one after the other **with replacement**. On pen and paper, construct a probability tree diagram and compute the probability of drawing one blue and one red ball (in any order).

17. Using the same bag as in the previous question, now assume that the two balls are drawn **without replacement**. On pen and paper, draw the probability tree diagram and calculate the probability of drawing one blue ball and one red ball in any order.

18. A teacher represents student exam outcomes using a probability tree. The probability that a student studies is 0.65; if they study, the probability of passing is 0.8, and if they do not study (probability 0.35), the probability of passing is 0.3. Compute the overall probability that a student passes the exam.

19. In your own words, explain why the probabilities on the branches emanating from any one node of a probability tree must sum to 1. Provide an example from a two-event tree diagram to support your explanation.

20. Describe, in a short paragraph, how probability trees can simplify solving probability problems for sequential events. Provide an example to illustrate your explanation.

Hard Questions

21. Construct a probability tree for three sequential events. The probabilities are as follows:

    - First event: $P(A) = 0.5$ and $P(B) = 0.5$.
    - Second event (if A occurs): $P(C \mid A) = 0.7$ and $P(D \mid A) = 0.3$; (if B occurs): $P(C \mid B) = 0.4$ and $P(D \mid B) = 0.6$.
    - Third event (if C occurs): outcomes E with probability 0.9 and F with probability 0.1; (if D occurs): outcomes E with probability 0.2 and F with probability 0.8.

    Using your diagram, calculate the probability of the sequence A, then C, then E.

22. Using the probability tree diagram you constructed in the previous question, compute the total probability that outcome E occurs at the third stage.

23. A game is modelled using a probability tree with three stages. In the first stage, a player chooses $Left$ with probability 0.4 and *Right* with probability 0.6. If $Left$ is chosen, the probability of winning (second stage) is 0.5, and if *Right* is chosen, the probability of winning is 0.7. In the third stage, if the player wins, they receive a bonus with probability 0.8 (and no bonus if they lose). Compute the overall probability that the player receives a bonus.

24. Consider a probability tree defined as follows. The first event produces outcomes X and Y with probabilities p and $1 - p$, respectively. If X occurs, then the outcomes are Z with probability 0.3 and W with probability 0.7. If Y occurs, then the outcomes are Z with probability q and W with probability $1 - q$. Given that the overall probability of obtaining Z is 0.5, show that

    $$p \cdot 0.3 + (1 - p) \cdot q = 0.5.$$

    Then, express q in terms of p and provide an example value for p and q that satisfy this equation.

25. Prove that in any probability tree the sum of the probabilities at the final level equals 1. Use an example of a three-event tree diagram to illustrate your proof.

26. In the following tree diagram, the first event splits into M and N with probabilities 0.6 and 0.4, respectively. After M, the outcomes are O with probability 0.5 and P with probability 0.5. After N, the outcomes are O with probability 0.2 and P with probability 0.8. Calculate the probability that the first event was M given

that outcome O occurred.
(Hint: Use the probabilities along the tree branches to form a ratio.)

27. A rare event is modelled by a probability tree with two stages. The probability of success in stage 1 is 0.1. If stage 1 is successful, the probability of success in stage 2 is 0.05; if stage 1 fails, the probability of success in stage 2 is 0.02. Compute:

    (a) the overall probability of success in stage 2, and

    (b) the probability that a stage 2 success originated from a stage 1 success.

28. Two bags are available. Bag A is selected with probability 0.4 and Bag B with probability 0.6. Bag A contains 2 green and 3 red balls, while Bag B contains 4 green and 1 red ball. One ball is drawn from the chosen bag. Draw a probability tree diagram on pen and paper and compute the overall probability of drawing a green ball.

29. Construct a probability tree for an experiment involving three sequential dependent events. Write a generic expression for the probability of any given outcome sequence in terms of the conditional probabilities on each branch. In your answer, briefly describe how you read the tree diagram to determine the overall probability.

30. Challenge: In an experiment, a probability tree is constructed where the probabilities are defined by the variable x. In the first stage, the branches are: branch A with probability x and branch B with probability $1 - x$. In the second stage, if A occurs, branch C has probability $x+0.2$ and branch D has probability $1-(x+0.2)$; if B occurs, branch C and branch D each have a probability of 0.5. Given that the overall probability of branch C is 0.6, form the equation

$$x \cdot (x + 0.2) + (1 - x) \cdot 0.5 = 0.6,$$

and solve for x. (Discard any solutions that are not valid probabilities.)