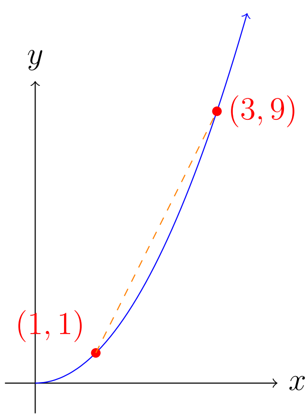
In this worksheet you will explore how to calculate both average and instantaneous rates of change and link these ideas with real-world situations. You will work through problems that reinforce the concept of the secant line (average rate) and the tangent line (instantaneous rate) using limits.

Easy Questions

1. Calculate the average rate of change of the function $f(x) = 3x + 2$ between $x = 1$ and $x = 5$. Write your answer as a fraction in simplest form.
2. Describe in your own words what is meant by the average rate of change of a function. Use a short sentence.
3. A car's distance traveled is given by $s(t) = 60t$ (where s is in kilometres and t in hours). Find the average speed between $t = 2$ and $t = 4$.
4. For $g(x) = x^2$, find the average rate of change between $x = 2$ and $x = 5$.
5. Explain why the average rate of change over any interval is the same for a linear function but may differ for a quadratic function.

Intermediate Questions

6. For $h(x) = 2x^2 - x$, determine the average rate of change between $x = 1$ and $x = 4$.
7. Given $f(x) = x^3$, calculate the average rate of change from $x = 1$ to $x = 3$.
8. Below is a diagram of the curve $y = x^2$ and the secant line joining $(1, 1)$ and $(3, 9)$. Using the diagram, determine the average rate of change between $x = 1$ and $x = 3$.



9. The temperature T (in degrees Celsius) at time t (in hours) is given by $T(t) = 20 + 2t$. Find the average rate of change of the temperature between $t = 0$ and $t = 5$.

10. A company's profit P (in dollars) is given by $P(x) = 50x - 300$, where x is the number of units sold. Explain what the average rate of change represents in this context.

11. Use the definition of the derivative to find the instantaneous rate of change of $f(x) = 4x + 1$ at any point x. (Hint: Compute $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.)

12. Describe the difference between the average rate of change and the instantaneous rate of change for a function. Use a short phrase.

13. A town's population is modelled by $P(t) = 1000 + 50t$, where t is measured in years. Determine the average rate of change of the population between $t = 2$ and $t = 8$ and explain its significance.

14. A car's position along a straight road is given by $s(t) = 5t^2 + 2t$ (with s in metres and t in seconds). Compute the average rate of change of the position between $t = 1$ and $t = 3$. What real-life quantity does this represent?

15. Let $f(x) = |x - 3|$. Compute the average rate of change between $x = 5$ and $x = 7$. (Assume x-values where the function is differentiable.)

16. The function f is given by the following table:

x	$f(x)$
2	10
4	18
6	26

Calculate the average rate of change between $x = 2$ and $x = 6$.

17. For $f(x) = x^2 + 3x$, use the limit definition (without applying any advanced differentiation rules) to find the instantaneous rate of change at $x = 2$.

18. A company's cost function is given by $C(x) = 200 + 15x$, where x represents the number of items produced. What does the average rate of change of $C(x)$ represent in a practical context?

19. Compute the instantaneous rate of change of $f(x) = 3x^2 - 2x$ at $x = 3$ using the definition $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.

20. On a graph of $y = f(x)$, a tangent line is drawn at $x = 4$. Explain how the slope of this tangent line is related to the instantaneous rate of change at $x = 4$.

Hard Questions

21. Prove that the instantaneous rate of change of any differentiable function $f(x)$ can be expressed as the limit $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Outline each step in your proof.

22. Consider the piecewise function

$$f(x) = \begin{cases} 2x+1 & \text{if } x < 3, \\ x^2-3 & \text{if } x \geq 3, \end{cases}$$

determine the instantaneous rate of change of $f(x)$ at $x = 2$ and justify why the limit definition applies at this point.

23. A graph shows a curve and a secant line between $x = 1$ and $x = 3$, and at $x = 2$ the tangent appears to be steeper than the secant. Explain, using the concepts of average and instantaneous rate of change, why this might occur.

24. An economics function is given by $C(x) = 1000 + 25x$, where x is the number of items produced. Compute the instantaneous rate of change at any x and interpret what this rate represents in the context of production costs.

25. If $f(x) = 2x + 3$, then its inverse function is $f^{-1}(x) = \frac{x-3}{2}$. Given that the instantaneous rate of change of f is 2, determine the instantaneous rate of change of f^{-1} and explain your reasoning.

26. Experimental data for a moving object results in the following position measurements: when $t = 1$, $s = 4$, and when $t = 3$, $s = 16$. Estimate the instantaneous rate of change at $t = 3$ by computing the average rate over progressively smaller intervals ending at $t = 3$. Explain your method.

27. For the function $f(x) = \sqrt{x}$, find an expression for the average rate of change between $x = a$ and $x = a + h$ in terms of a and h. Then, discuss the behaviour as $h \to 0$.

28. Show in detail how the limit $\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$ simplifies to yield the instantaneous rate of change for $f(x) = x^2$. Explain every algebraic step.

29. A table of average rates was computed for the function $f(x) = x^2$ at $x = 3$ using intervals $[3, 3+h]$. The values obtained for $h = 1$, 0.5, and 0.25 are 7, 6.5, and 6.25 respectively. Use these data to estimate the instantaneous rate of change at $x = 3$ and justify your estimation.

30. A cyclist's distance travelled (in kilometres) is given by $s(t) = 4t^2 + 2t$, where t is in hours.

    (a) Calculate the average rate of change in position between $t = 1$ and $t = 3$.

    (b) Using the limit definition, compute the instantaneous rate of change at $t = 2$.

    (c) On a piece of paper, sketch the graph of $s(t)$ and draw the secant line between $t = 1$ and $t = 3$, and the tangent line at $t = 2$. Label clearly.

    Provide a brief explanation for each part.