ILLAWARRA TUTORING

This worksheet focuses on exploring how to calculate both average and instantaneous rates of change and linking these ideas to real-world situations. You will practice computing the average change of a function over an interval and explore the concept of the instantaneous rate of change, which represents the limit of the average rate as the interval becomes infinitesimally small.

Easy Questions

1. Please calculate the average rate of change of the function $f(x) = 2x + 3$ between $x = 1$ and $x = 4$.
2. Compute the average rate of change of $f(x) = x^2$ on the interval $[1, 3]$.
3. State the average and instantaneous rate of change for the constant function $f(x) = 5$ between $x = 2$ and $x = 6$.
4. Calculate the average rate of change of $f(x) = 3x^2 + 2x - 1$ from $x = -1$ to $x = 2$.
5. Write down the definition of the instantaneous rate of change in your own words.

Intermediate Questions

6. For $f(x) = x^2$, derive an expression for the average rate of change between $x = 2$ and $x = 2 + h$. Then, explain what happens as $h \to 0$.
7. A town's population is modelled by $P(t) = 1000 + 50t$, where t represents time in years. Calculate the average rate of change of the population over a period of 5 years and explain what this rate represents.
8. A particle moves along a line so that its distance (in metres) is given by $s(t) = 4t^2 + 2t$, where t is in seconds. Find its average speed between $t = 1$ and $t = 3$ seconds.
9. For the linear function $f(x) = 3x + 7$, show that its instantaneous rate of change is equal to its average rate of change over any interval.
10. Determine the average rate of change of $f(x) = x^2 + 4x + 1$ between $x = -2$ and $x = 0$.
11. Calculate the average rate of change of $f(x) = \sqrt{x + 9}$ from $x = 0$ to $x = 7$.
12. Explain, in plain text, the graphical meaning of the instantaneous rate of change of a function at a point.

13. For the function $f(x) = \frac{1}{x}$, compute the average rate of change between $x = 1$ and $x = 2$.

14. A graph passes through the points $(2, 7)$ and $(5, 19)$. Calculate its average rate of change.

15. The function $f(t) = t^3$ models a certain volume. Compute the average rate of change of this function between $t = 1$ and $t = 2$.

16. Determine the average rate of change of $g(x) = \sin(x)$ on the interval $\left[0, \frac{\pi}{2}\right]$.

17. For $h(x) = |x|$, find the average rate of change over the interval $[-3, 3]$.

18. A car's position is given by $s(t) = t^2 + 3t + 2$, where t is in seconds. Compute the instantaneous speed of the car at $t = 4$ seconds by considering the limit of the average rate of change.

19. A projectile's height is given by $h(t) = 100t - 5t^2$. Calculate the average rate of change of the height between $t = 0$ and $t = 4$ seconds.

20. Describe, in a short paragraph, how the graph of a function might look if its instantaneous rate of change is increasing.

Hard Questions

21. Using the limit definition of the instantaneous rate of change, find the instantaneous rate of change of $f(x) = x^2$ at $x = 3$.

22. For $f(x) = 2x^2 + 3x$, determine the instantaneous rate of change at $x = 1$ by computing the limit of the average rate of change as the interval shrinks to zero. Then, discuss the significance of this value.

23. Consider the function $f(x) = \frac{1}{x+2}$. Derive an expression for the average rate of change between x and $x + h$, and use this to find the instantaneous rate of change at $x = 0$.

24. A function $f(x) = x^2 + 10x + 25$ represents the cost (in dollars) of producing x units. Find the instantaneous rate of change when $x = 5$, and explain what this rate of change indicates in an economic context.

25. The temperature (in degrees Celsius) is given by $T(t) = 20 + 5t - \frac{t^2}{2}$ for $0 \leq t \leq 8$. Compute (a) the average rate of change over the first 4 hours and (b) the instantaneous rate of change at $t = 4$. Compare and explain any differences between the two rates.

26. Describe and sketch, using pen and paper, a possible graph of a function that exhibits a decreasing average rate of change yet has an interval where the instantaneous rate of change is increasing. In your answer, explain your reasoning.

27. For the function $f(x) = \sqrt{x}$ (with $x \geq 0$), derive an expression for the average rate of change between x and $x + h$. Simplify your answer and discuss how the average rate changes as x increases.

28. A company's profit is described by $P(x) = -2x^2 + 120x - 1000$, where x represents the number of items produced. Determine the average rate of change in profit as production increases from $x = 20$ to $x = 30$. Then, estimate the instantaneous rate of change at $x = 25$, commenting on its business implications.

29. In an experiment, the concentration of a chemical is given by $C(t) = 0.5t^2 + 3t + 50$, where t is measured in minutes. (a) Calculate the average rate of change of $C(t)$ between $t = 2$ and $t = 5$. (b) Estimate the instantaneous rate of change at $t = 3$ by considering a very small interval around $t = 3$.

30. Suppose a function f is differentiable at $x = a$. If the average rate of change over a very small interval around a is 7, what is the instantaneous rate of change at $x = a$? Explain the relationship between these rates.