This worksheet focuses on differentiability. You will learn to determine where a function is differentiable and why this property is important. Read each question carefully and provide clear, reasoned answers.

Easy Questions

1. Write a short definition of what it means for a function to be differentiable at a point.

2. Determine whether the function $f(x) = 3x+2$ is differentiable at every point. State your reasoning.

3. Decide if the function $f(x) = x^2$ is differentiable at $x = 5$. Explain your answer.

4. Determine whether the function $f(x) = |x|$ is differentiable at $x = 0$. Give a brief explanation.

5. Sketch the graph of the function $f(x) = |x|$ using pen and paper. Mark clearly the point at which the function is not differentiable and explain why this point causes a problem for differentiability.

Intermediate Questions

6. Consider the function

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 2, \\ 4x - 4, & \text{if } x > 2. \end{cases}$$

Determine if f is differentiable at $x = 2$. Provide detailed working.

7. Let
$$f(x) = \begin{cases} \sin x, & \text{if } x \le \frac{\pi}{2}, \\ 1, & \text{if } x > \frac{\pi}{2}. \end{cases}$$
Determine whether f is differentiable at $x = \frac{\pi}{2}$ and justify your conclusion.

8. Explain why continuity at a point is a necessary, but not sufficient, condition for differentiability.

9. Determine the differentiability of the function $f(x) = x^{\frac{1}{3}}$. Consider whether there are any points where the derivative may not exist.

10. Analyse the function $f(x) = \sqrt{x}$ for $x \ge 0$. Identify the points at which f is differentiable and explain your reasoning.

11. Consider $f(x) = x^{\frac{2}{3}}$. Determine if f is differentiable at $x = 0$. Provide reasons for your answer.

12. Given
$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$
determine if f is differentiable at $x = 0$.

13. Using the definition of differentiability, find the derivative at $x = 1$ for $f(x) = 2x^2 - 3x + 1$. Show your working.

14. Let $f(x) = |x - 1|$. Determine if f is differentiable at $x = 1$, stating your reasoning.

15. Consider the function
$$f(x) = \begin{cases} x^2, & \text{if } x < 0, \\ x, & \text{if } x \ge 0. \end{cases}$$
Determine whether f is differentiable at $x = 0$ and explain your answer.

16. Sketch the graph of $f(x) = |x-2| + 1$ using pen and paper. Identify any point at which the function is not differentiable.

17. Provide an example of a function that is not continuous at a point and explain why it fails to be differentiable there.

18. Explain the relationship between differentiability and the existence of a tangent line at a point.

19. For the function
$$f(x) = \begin{cases} 3x+1, & \text{if } x \leq 4, \\ x^2 - 5, & \text{if } x > 4, \end{cases}$$
determine whether f is differentiable at $x = 4$. Explain the conditions that must hold for differentiability at this point.

20. Consider the function $f(x) = \sqrt{|x|}$. Determine the points at which f is differentiable and give reasons for your conclusion.

Hard Questions

21. Let
$$f(x) = \begin{cases} ax+b, & \text{if } x < 2, \\ cx^2, & \text{if } x \geq 2. \end{cases}$$
Find the values of a, b, and c so that f is differentiable at $x = 2$.

22. Prove that if a function f is differentiable at $x = c$, then f is continuous at $x = c$. State each step clearly.

23. Find the necessary and sufficient conditions on the parameters p and q so that
$$f(x) = \begin{cases} x^2 + p, & \text{if } x \leq 1, \\ qx + 1, & \text{if } x > 1, \end{cases}$$
is differentiable at $x = 1$.

24. Determine if the function

$$f(x) = \begin{cases} \dfrac{\sin(x)}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0, \end{cases}$$

is differentiable at $x = 0$. Provide a clear explanation.

25. For the function

$$f(x) = \begin{cases} x \sin \dfrac{1}{x}, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

demonstrate that f is differentiable at $x = 0$.

26. Consider

$$f(x) = \begin{cases} x^3, & \text{if } x \leq 1, \\ 3x - 2, & \text{if } x > 1. \end{cases}$$

Determine whether f is differentiable at $x = 1$ and justify your answer.

27. Define the function

$$f(x) = \begin{cases} x \ln(x), & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$$

Discuss the differentiability of f at $x = 0$ with proper reasoning.

28. Show that the function $f(x) = x|x|$ is differentiable everywhere. Include justification using the definition of the derivative.

29. Let

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0, \\ -\sqrt{-x}, & \text{if } x < 0. \end{cases}$$

Determine whether f is differentiable at $x = 0$ and explain your rationale.

30. Discuss why the presence of a sharp corner or cusp in the graph of a function prevents differentiability at that point. Provide at least one example to illustrate your explanation.