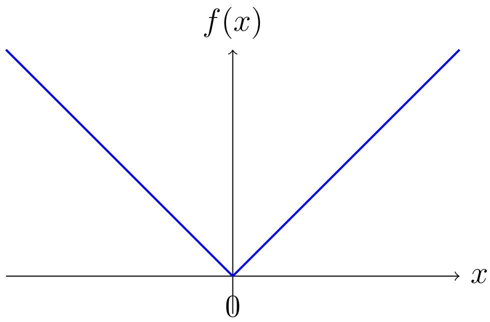
ILLAWARRA TUTORING

In this worksheet you will learn how to determine where a function is differentiable and why this property is important. You will analyse functions using the limit definition of the derivative and explore various examples, including piecewise functions and functions with corners or cusps.

Easy Questions

1. Write a short explanation of why the function $f(x) = x^2$ is differentiable for every real number. Include a brief discussion of the limit definition of the derivative.

2. Sketch the graph of $f(x) = |x|$ using the diagram provided below and state at which point the function is not differentiable. Explain your reasoning.



3. Explain why the constant function $f(x) = 5$ is differentiable for all real numbers. State what its derivative is at any point.

4. Using the limit definition of the derivative, show that $f(x) = x$ is differentiable at $x = 1$. Provide all steps in your explanation.

5. Determine whether the function $f(x) = \sqrt{x}$ (with domain $x \geq 0$) is differentiable at $x = 0$. Explain your answer using the definition of differentiability.

Intermediate Questions

6. Prove that if a function f is differentiable at a point $x = a$, then f is continuous at $x = a$. Write your proof using the limit definition of the derivative.

7. Consider the function

$$f(x) = \begin{cases} x^2, & x \leq 2, \\ 4x - 4, & x > 2. \end{cases}$$

Determine whether f is differentiable at $x = 2$. Justify your answer by checking the left-hand and right-hand derivatives.

8. For the function
$$f(x) = \begin{cases} x, & x < 1, \\ 2 - x, & x \geq 1, \end{cases}$$
explain whether f is differentiable at $x = 1$. Include in your answer a discussion of the slopes of the lines on either side of $x = 1$.

9. Find all points where the function $f(x) = |x - 3| + 2$ is not differentiable. Explain how the structure of the function leads to these non-differentiable points.

10. Investigate the differentiability of $f(x) = x^{1/3}$ at $x = 0$. Use the limit definition of the derivative to support your answer.

11. Analyse the function $f(x) = \sqrt{x}$ (with $x \geq 0$). Determine for which values of x the function is differentiable, paying particular attention to $x = 0$. Explain your reasoning.

12. Explain why the function $f(x) = x^{\frac{2}{3}}$ is not differentiable at $x = 0$. In your explanation, compute the one-sided limits of the difference quotient.

13. Show that the function
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$
is differentiable at $x = 0$. Use the limit definition of the derivative and discuss how the factor x^2 influences the differentiability.

14. Consider the function
$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$
Determine whether f is differentiable at $x = 0$. Explain your analysis clearly.

15. Find the points at which the function
$$f(x) = \begin{cases} x^2, & x \leq 0, \\ -x^2, & x > 0, \end{cases}$$
is differentiable. Provide detailed reasoning based on the limits from either side of the potential non-differentiable point.

16. Explain why a sharp corner, such as the one in the graph of $f(x) = |x|$, leads to non-differentiability at that point. Use the idea of unequal left-hand and right-hand slopes in your explanation.

17. Consider the function

$$f(x) = \begin{cases} 2x+1, & x < -2, \\ x^2, & -2 \le x \le 2, \\ 3x-1, & x > 2. \end{cases}$$

Determine whether f is differentiable at $x = -2$ and $x = 2$. Justify your conclusion by computing the appropriate one-sided derivatives.

18. Use the limit definition of the derivative to show that the function $f(x) = x^3$ is differentiable at any point $x = a$. Provide all necessary steps in your calculation.

19. Prove that if a function f is differentiable at $x = a$, then the tangent line to f at $(a, f(a))$ exists. Explain the connection between the derivative and the slope of the tangent line.

20. Provide an example of a function that is continuous at a point but not differentiable there. Use the limit definition of the derivative to explain precisely why the derivative does not exist at that point.

Hard Questions

21. Consider the function

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that f is not differentiable at $x = 0$ by analysing the limit of the difference quotient.

22. Demonstrate that the function $f(x) = x^{\frac{2}{3}}$ is not differentiable at $x = 0$. Include an examination of the one-sided limits of the difference quotient in your explanation.

23. Analyse the differentiability at $x = 0$ of the function

$$f(x) = \begin{cases} x + x^{\frac{2}{3}}\sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Determine whether the derivative exists at $x = 0$ and explain your reasoning.

24. Let

$$f(x) = \begin{cases} x^2, & x \text{ is rational}, \\ 0, & x \text{ is irrational}. \end{cases}$$

Examine the differentiability of f at $x = 0$. Provide a detailed explanation using the limit definition.

25. Consider the function

$$f(x) = \begin{cases} x^2, & x \text{ is rational}, \\ 2x^2, & x \text{ is irrational}. \end{cases}$$

Determine whether f is differentiable at $x = 0$. Explain the role played by the differing definitions in your answer.

26. Show that the function

$$f(x) = \begin{cases} x^3 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is differentiable at $x = 0$. Provide a rigorous justification using the limit definition.

27. Investigate the differentiability of the function $f(x) = |x - 1| + |x + 1|$. Identify all points where f is not differentiable and explain why these points cause a breakdown in differentiability.

28. Let $f(x) = \max x, x^2$. Determine the set of points at which f is differentiable. Provide an explanation based on the definition of f and the properties of maximum functions.

29. Consider the function $f(x) = \min x, x^2$. Identify all points where f is differentiable and justify your answer by discussing the derivatives from either side of the boundary where the minimum switches.

30. Write a short discussion on the importance of differentiability in understanding the behaviour of functions. In particular, explain how differentiability relates to the existence of a unique tangent line at a point and its geometric implications.