ILLAWARRA TUTORING

In this worksheet you will learn how to determine where a function is differentiable and why this property is important. You will explore definitions, work through piecewise examples and prove results relating to differentiability. Remember that a function is differentiable at a point if the derivative exists there; this requires that the function is continuous at the point and that the left-hand and right-hand limits of the difference quotient agree.

Easy Questions

1. Write a clear definition of what it means for a function $f(x)$ to be differentiable at a point a.

2. Consider the function $f(x) = x^2$. State whether f is differentiable for all $x \in \mathbb{R}$ and briefly explain why.

3. Explain in your own words why differentiability at a point implies continuity at that point.

4. Determine if $f(x) = 3x + 2$ is differentiable on $\mathbb{R}$ and provide a brief explanation.

5. Consider the function $f(x) = |x|$. State whether f is differentiable at $x = 0$ and explain your reasoning.

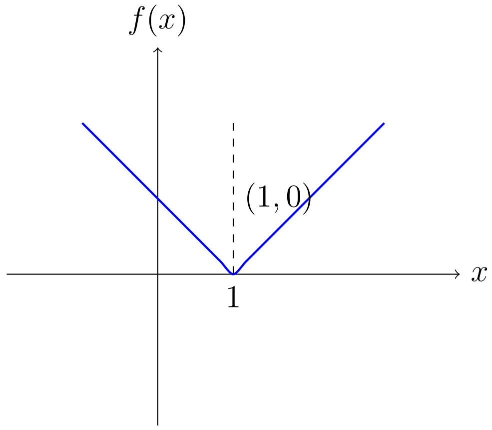
Intermediate Questions

6. Consider

$$f(x) = \begin{cases} x^2, & \text{if } x \leq 2, \\ 4x - 4, & \text{if } x > 2. \end{cases}$$

Determine whether f is differentiable at $x = 2$, showing that the left-hand and right-hand derivatives agree.

7. Consider $f(x) = |x - 1|$. Using the diagram below, determine whether f is differentiable at $x = 1$.



8. Explain why a function must be continuous at a point to be differentiable there.

9. Prove that if a function f is differentiable at a, then f is continuous at a.

10. Consider
$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ -x^2, & \text{if } x < 0. \end{cases}$$
Compute the left-hand and right-hand derivatives at $x = 0$ and state whether f is differentiable at 0.

11. Sketch a function that is continuous everywhere but not differentiable at one point. Provide a brief explanation of your sketch.

12. For $f(x) = |x|$, calculate the left-hand derivative and the right-hand derivative at $x = 0$ and explain why they demonstrate that f is not differentiable at 0.

13. Let
$$f(x) = \begin{cases} ax + b, & \text{if } x \leq 1, \\ x^2, & \text{if } x > 1. \end{cases}$$
Find the values of a and b that make f continuous and differentiable at $x = 1$.

14. Let
$$f(x) = \begin{cases} x^2, & \text{if } x < 3, \\ kx - 2, & \text{if } x \geq 3. \end{cases}$$
Determine the value of k required for f to be differentiable at $x = 3$.

15. Consider $f(x) = x^{\frac{1}{3}}$. Investigate the differentiability of f at $x = 0$ using the limit definition.

16. Consider $f(x) = x|x|$, which can be written as

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0, \\ -x^2, & \text{if } x < 0. \end{cases}$$

Determine if f is differentiable at $x = 0$.

17. Let $f(x) = \min(x^2, 4)$. Determine the points in $\mathbb{R}$ at which f is differentiable.

18. Consider

$$f(x) = \begin{cases} \sin x, & \text{if } x \leq 0, \\ 2x, & \text{if } x > 0. \end{cases}$$

Discuss the differentiability of f at $x = 0$.

Hard Questions

21. Consider

$$f(x) = \begin{cases} x^3 + cx^2, & x < 1, \\ ax + b, & x \geq 1. \end{cases}$$

Find the values of a, b, and c so that f is continuous and differentiable at $x = 1$.

22. Let

$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Prove that f is continuous at $x = 0$ and determine whether f is differentiable at 0.

23. Define

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that f is differentiable at $x = 0$ even though f' is not continuous at 0.

24. For

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

compute $f'(0)$ using the limit definition and discuss the differentiability at 0.

25. Let

$$f(x) = \begin{cases} \sin x - x, & x \leq 0, \\ 2x, & x > 0. \end{cases}$$

Determine whether f is differentiable at $x = 0$.

26. Consider
$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$
Investigate the differentiability of f at $x = 0$.

27. Provide a rigorous proof that a function may be continuous at a point but not differentiable there. Use $f(x) = |x|$ as an example.

28. Let
$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q}, \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
Determine the points at which f is differentiable.

29. Prove using the epsilon-delta definition that if f is differentiable at a, then the limit
$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
exists and is unique from both sides.