This worksheet explores key properties of functions such as continuity, increasing and decreasing behaviour, and symmetry. Work carefully through the questions and show your reasoning where appropriate.

Easy Questions

1. Write in your own words what it means for a function to be continuous at a point.
2. Consider the function $f(x) = x^2$. Is this function continuous for all real numbers? Give a brief explanation.
3. For the function $f(x) = x^3$, state whether the function is increasing, decreasing, or neither over its entire domain.
4. Identify whether the function $f(x) = x^2$ is symmetric. If it is, specify the type of symmetry it has.
5. Decide whether the function $f(x) = x^3$ is even, odd, or neither and explain briefly.

Intermediate Questions

6. Consider the function $f(x) = \dfrac{1}{x}$. State its continuity properties and explain why it is not continuous for all real numbers.
7. Consider the piecewise function

$$f(x) = \begin{cases} x+1 & \text{if } x < 0, \\ x^2 & \text{if } x \geq 0, \end{cases}$$

and investigate whether f is continuous at $x = 0$. Support your answer with a brief explanation.

8. Recall that a function f is increasing on an interval if for any $x_1 < x_2$ in that interval, $f(x_1) < f(x_2)$. Examine the function $f(x) = x^2$ and state on which of the intervals $(-\infty, 0]$ and $[0, \infty)$ the function is increasing or decreasing.
9. State the definition of an odd function in terms of its symmetry.
10. Give an example of a function that is both continuous and symmetric (either even or odd). Include a brief justification for your example.
11. Provide the definition of a function being decreasing on an interval.

12. Consider $f(x) = |x|$. Discuss its continuity and symmetry properties.

13. For the function $f(x) = \sqrt{x}$, state its domain and discuss whether it is continuous on its domain.

14. For the function $f(x) = \dfrac{x}{1+x^2}$, determine whether it is even, odd, or neither. Explain your reasoning.

15. Consider $f(x) = \sin x$. Investigate its continuity and state whether it is increasing or decreasing on the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

16. Describe what it means for a function to have a turning point (local maximum or minimum) in relation to its increasing or decreasing behaviour.

17. Consider the function $f(x) = 2x^2 - 4x + 1$. Without calculating the vertex, explain how its graph displays symmetry.

18. Define the term local maximum and provide an example of a function (written in formula form) that exhibits a local maximum.

19. For the function $f(x) = \dfrac{1}{x^2}$, discuss its continuity (state where it is defined) and determine its symmetry.

20. Explain how the concept of continuity of a function is related to the Intermediate Value Theorem.

Hard Questions

21. Consider $f(x) = \dfrac{x^2 - 1}{x - 1}$. Discuss the continuity of this function at $x = 1$ and explain your reasoning.

22. Define the function

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0, \end{cases}$$

and investigate its continuity at $x = 0$ using limit arguments.

23. For the function $f(x) = e^x$, determine the intervals on which the function is increasing or decreasing. Provide a brief justification for your answer.

24. Consider the function $f(x) = x^{\frac{1}{3}}$. Discuss its continuity and monotonicity properties.

25. The function

$$f(x) = \begin{cases} x + 2 & \text{if } x < -1, \\ x^2 & \text{if } -1 \leq x \leq 2, \\ 3x - 4 & \text{if } x > 2, \end{cases}$$

is defined piecewise. Investigate the continuity of f at $x = -1$ and $x = 2$.

26. Define and explain the concept of symmetry for a function that is neither even nor odd.

27. Sketch the graph of $f(x) = x^3 - 3x$ on a sheet of paper. Then, identify the intervals where the function appears to be increasing and where it is decreasing. Indicate any turning points you observe.

28. For the function $f(x) = \frac{2x}{x^2+1}$, analyse its symmetry. State whether it is even, odd, or neither, and justify your answer.

29. Suppose a function f satisfies $f(2-x) = f(2+x)$ for every x in its domain. Discuss what this condition reveals about the symmetry of f and provide an example to illustrate your explanation.

30. Let $f(x) = \frac{x^2}{x^2+1}$. Discuss the continuity of f (state its domain) and, without using calculus techniques, explain the increasing or decreasing behaviour of f over the real numbers.