ILLAWARRA TUTORING

In this worksheet you will develop an understanding of the significance of Euler's number, e, and its role in exponential functions. You will explore its limit definition, its series expansion, and appreciate some of its unique properties.

Easy Questions

1. Write down the value of e correct to three decimal places.
2. In a short sentence, state why e is important in modelling continuous growth.
3. Write the limit definition of e, expressing it as $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.
4. Estimate the value of $\left(1+\frac{1}{10}\right)^{10}$.
5. Simplify the expression e^0.

Intermediate Questions

6. Using the limit definition of e, calculate an approximation by taking $n = 20$. Write your answer to four decimal places.
7. Repeat the calculation from the previous question with $n = 100$. Compare your result with that from question 6 and state your observation.
8. Write a brief explanation of what the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ represents in relation to e.
9. Write down the first four terms (from $n = 0$ to $n = 3$) of the series expansion for e, given by $e = \sum_{n=0}^{\infty}\frac{1}{n!}$.
10. Evaluate the sum $1 + 1 + \frac{1}{2} + \frac{1}{6}$ and use this to give an approximate value for e.
11. In a short paragraph, explain how the series expansion $\sum_{n=0}^{\infty}\frac{1}{n!}$ provides a method of approximating e.

12. Describe one unique mathematical property of e that distinguishes it from other numbers in the context of exponential functions.

13. Calculate the sum $\sum_{n=0}^{4} \frac{1}{n!}$. Write your answer as a decimal rounded to five decimal places.

14. Compare the approximations of e obtained by using the limit definition with $n = 10$ and the series expansion up to $n = 3$. Briefly state which method seems to give a closer approximation and why.

15. In one or two sentences, explain what the number e represents in the context of modelling processes with continuous growth.

16. Verify that $e^0 = 1$ by explaining, in a sentence, the property of the exponential function that leads to this result.

17. In a short paragraph, explain how e arises naturally in contexts of continuous compounding, such as in certain financial models, without using any calculus notation.

18. Write an expression using e which represents continuous growth at a constant rate r over time t, without solving or simplifying further.

19. Consider the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$. Write a short explanation describing why this sequence converges to e.

20. Provide one example of a real-world phenomenon where the number e plays an important role. Describe this phenomenon in a short paragraph.

Hard Questions

21. Derive the series expansion for e by considering the Taylor series expansion of $\exp(x)$ at $x = 0$ and then evaluating at $x = 1$. Clearly state each step in your derivation.

22. Prove that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. Outline your arguments based on a recognised convergence test.

23. For the series expansion of e, derive an inequality that can be used to estimate the error (remainder) after summing the first N terms.

24. Evaluate the limit $\lim_{n\to\infty} n \left[\left(1 + \frac{1}{n}\right) - e^{\frac{1}{n}}\right]$. Provide a detailed explanation of your approach.

25. Explain in a short essay the relationship between the limit definition $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ and the series expansion $e = \sum_{n=0}^{\infty} \frac{1}{n!}$. Discuss why both definitions are equivalent.

26. Outline a proof that shows e is irrational using its series expansion. Include the main steps and justifications.

27. Consider the sequence $a_n = \left(1 + \frac{1}{n}\right)^n$. Derive an expression or inequality that describes the rate at which a_n converges to e. Explain your reasoning.

28. Derive an expression for the error term when approximating e using its series expansion sum up to $\frac{1}{N!}$. Explain how the next term in the series provides a bound for the error.

29. Compare the approximation methods of using the limit definition $\left(1 + \frac{1}{n}\right)^n$ and the series expansion $\sum_{n=0}^{N} \frac{1}{n!}$. Discuss the advantages and disadvantages of each approach in a short essay.

30. Using the series expansion for the exponential function, derive a formula for e^k, where k is a constant, expressed as a series. In your answer, discuss the significance of this representation in the context of exponential growth.