ILLAWARRA TUTORING

In this worksheet you will understand the significance of Euler's number e and its role in exponential functions. You will explore its definitions via both limits and series, examine its numerical approximations and convergence properties, and discuss its importance in modelling continuous growth.

Easy Questions

1. Write the approximate value of e to four decimal places.
2. Write the limit definition of Euler's number as a sequence.
3. Write the series expansion for e as an infinite sum.
4. Calculate $\left(1+\frac{1}{1}\right)^1$ and $\left(1+\frac{1}{2}\right)^2$.
5. Name one real-life application where Euler's number is used to model continuous growth.

Intermediate Questions

6. Explain in a short paragraph how e is used to model continuous compound interest.
7. Compute $\left(1+\frac{1}{n}\right)^n$ for $n = 5$, $n = 10$, and $n = 20$. Comment on its approach to e.
8. Estimate the value of e by evaluating $\left(1+\frac{1}{50}\right)^{50}$.
9. Write down the first five terms of the series expansion for e and compute their sum.
10. Explain why the infinite series for e converges.
11. Using your answer from question 9, estimate the error when approximating e with these five terms.
12. Write an expression for the remainder term when approximating e by the series up to n terms, and describe its behaviour as n increases.
13. Using the limit definition $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$, explain why e is an irrational number.

14. Confirm that $\left(1+\frac{1}{1000}\right)^{1000}$ is approximately 2.718 by performing a calculation.

15. List three different areas in mathematics or science where e plays an important role.

16. Using pen and paper, sketch the graph of $y=\left(1+\frac{1}{n}\right)^{n}$ for n from 1 to 30, and indicate its trend as n increases.

17. Explain in one or two sentences how the definition $\left(1+\frac{1}{n}\right)^{n}$ relates to the formula for continuously compounded interest.

18. Name one combinatorial interpretation or probabilistic scenario where e naturally occurs.

19. Calculate $\left(1+\frac{1}{10}\right)^{10}$ accurate to three decimal places.

20. Write a short paragraph discussing the importance of Euler's number in modelling natural exponential growth.

Hard Questions

21. Provide a proof outline showing that e is irrational, referring to the series expansion and a contradiction argument.

22. Show that the series for e converges much faster than the harmonic series by comparing the decay of their terms.

23. Derive an inequality that gives an upper bound for the error (remainder) when approximating e using the first $n+1$ terms of its series.

24. Prove that the sequence $\left(1+\frac{1}{n}\right)^{n}$ is increasing for positive integers n and that it is bounded above by e.

25. Demonstrate how the limit definition $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$ can be generalised when considering continuous compounding. Provide clear reasoning.

26. Using the series expansion for e, show that the remainder term is less than $\frac{1}{(n+1)!}$ for $n\geq 0$, and clearly derive the corresponding error bound.

27. With mathematical rigour, explain why e cannot be written as a ratio of two integers.

28. Determine the asymptotic rate at which $\left(1+\frac{1}{n}\right)^{n}$ converges to e as $n\to\infty$, and justify your result.

29. Prove that the limit definition $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ is equivalent to the series definition $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ by justifying the interchange of limit and summation.

30. Discuss in detail how Euler's number is used in the derivation of the continuous compounding formula in finance, and derive the formula $A = P \cdot e^{rt}$.