In this worksheet, you will learn about the significance of Euler's number e and its role in exponential functions. You will explore its definition via a limit, investigate its approximation through sequences and series, and discuss its importance in continuous growth processes.

Easy Questions

1. Please state the approximate value of e to three decimal places.
2. Write the limit definition of e as a limit of a sequence.
3. Evaluate $\left(1+\frac{1}{100}\right)^{100}$ approximately and compare it with the value of e.
4. Explain in your own words why e is regarded as a natural base for modelling growth processes.
5. Write the infinite series expression for e up to the first four terms.

Intermediate Questions

6. Calculate $\left(1+\frac{1}{10}\right)^{10}$ and comment on how close it is to e.
7. Compute $\left(1+\frac{1}{50}\right)^{50}$ and explain the trend as the denominator increases.
8. Describe why the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$ is relevant in the context of continuous compounding interest.
9. Restate the definition of e using the limit $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}$ and discuss its meaning.
10. Write the Maclaurin series expansion for e using the first three non-zero terms.
11. Evaluate the sum $1+\frac{1}{1!}+\frac{1}{2!}$ and compare it with e.
12. Compare $\left(1+\frac{1}{n}\right)^{n}$ with the series approximation $\sum_{k=0}^{2}\frac{1}{k!}$ for an appropriate value of n.

13. Explain why e is considered the natural base for exponential functions.

14. Create a table of values for $\left(1+\frac{1}{n}\right)^n$ using $n = 1, 2, 5, 10, 20$ and discuss how these values approach e.

15. Discuss why the sequence $\left(1+\frac{1}{n}\right)^n$ is increasing and bounded, and the significance of these properties in convergence.

16. Explain the difference between continuous compounding and discrete compounding, and the role that e plays in continuous compounding.

17. Compute $\left(1+\frac{1}{1000}\right)^{1000}$ to a suitable degree of accuracy and explain its relation to e.

18. Describe how increasing n in $\left(1+\frac{1}{n}\right)^n$ influences the approximation to e.

19. In your own words, explain the meaning of e being irrational and its implications for its decimal expansion.

20. Summarise in a short paragraph the key role of e in exponential functions and natural growth models.

Hard Questions

21. Derive the infinite series expansion for e starting from its limit definition, and explain each step of your derivation.

22. Provide a proof that the sequence $\left(1+\frac{1}{n}\right)^n$ is increasing as n increases.

23. Estimate the error when approximating e by $\left(1+\frac{1}{50}\right)^{50}$ and discuss the factors affecting this error.

24. Compare the approximations for e obtained from the limit $\left(1+\frac{1}{n}\right)^n$ (for a large n) and from the series $\sum_{k=0}^{3}\frac{1}{k!}$; discuss the advantages and drawbacks of each method.

25. Discuss why the series $\sum_{k=0}^{\infty}\frac{1}{k!}$ for e converges rapidly, including an explanation based on the growth of factorial terms.

26. Explain how the irrationality of e affects its decimal representation and why no repeating pattern occurs.

27. Investigate the rate of convergence of $\left(1+\frac{1}{n}\right)^n$ to e by calculating its value for $n = 10, 100, 1000$, and discuss your findings.

28. Determine how many terms in the series $\sum_{k=0}^{\infty} \frac{1}{k!}$ are required in order to approximate e with an error less than 0.001.

29. Using a numerical method, approximate e to three decimal places via both the limit definition $\left(1+\frac{1}{n}\right)^n$ for a large n and the series $\sum_{k=0}^{m} \frac{1}{k!}$; compare your results.

30. Write a short essay on the historical development of e, including its discovery by Euler, and discuss its significance in mathematics.