In this worksheet you will explore applications of exponential functions in modelling real-world scenarios such as compound interest, population growth, and radioactive decay. You will work through a series of questions that range from simple to challenging to help you understand how exponential models are formed and used.

Easy Questions

1. Write the compound interest formula for an initial principal P_0, an annual interest rate r, and time t (in years), when interest is compounded annually.
2. A radioactive substance decays such that its mass is reduced to half every 4 years. Write an exponential expression for the remaining mass $m(t)$ after t years in terms of the initial mass m_0.
3. A town has an initial population P_0. Its population increases at a constant rate such that after t years the population is given by an exponential function. Write this model if the population increases by 3% each year.
4. Evaluate the amount given by the model $A = P_0(1+r)^t$ when $P_0 = 1000$, $r = 0.05$, and $t = 2$.
5. In the exponential model $Q(t) = Q_0 a^t$, explain in plain language the meaning of the parameters Q_0 and a in a context of population growth.

Intermediate Questions

6. An investment of P_0 dollars grows at an annual rate of r compounded annually. Write the exponential model for the value of this investment after t years.
7. An investor deposits 2000 dollars in a bank that pays 6% interest per annum compounded annually. Compute the value of the investment after 3 years using the model $A = P_0(1 + r)^t$.
8. Write an exponential model for the amount A accumulated after t years if P_0 dollars are invested at an annual nominal interest rate of r compounded monthly.
9. For an account with an annual nominal rate r compounded monthly, express how to calculate the effective annual rate using an exponential model.
10. A 10 gram radioactive sample decays according to the model $m(t) = 10\left(\frac{1}{2}\right)^{t/4}$. Calculate the remaining mass after 8 years.

11. Show how to write the exponential decay model $m(t) = m_0 e^{kt}$ given that the half-life of the substance is $T_{1/2}$. Express k in terms of $T_{1/2}$.

12. A town's population grows at a steady rate of 2% per annum. Write an exponential model for the population after t years in terms of the initial population P_0.

13. For a population growing exponentially with a growth factor a, derive the expression for the doubling time in terms of a.

14. Given a population model $P(t) = P_0(1.04)^t$ and that the population after 5 years is 1216, determine the original population P_0.

15. A bacteria colony doubles every 7 hours. Write the exponential model for the number of bacteria $N(t)$ after t hours, and calculate the factor by which the population increases after 21 hours.

16. An investment grows according to $A = P_0(1 + r)^t$, where t is in months. If an investment of $P_0 = 5000$ dollars grows to 6000 dollars in 12 months, determine the monthly growth rate r.

17. A substance decays so that after 5 years only 30% of the initial amount remains. Write an exponential decay model in the form $A(t) = A_0(1 - d)^t$ and find the decimal value of d.

18. Using the decay model from Question 17, determine the half-life of the substance.

19. Write the compound interest formula for an investment compounded quarterly. Express the model clearly in terms of P_0, annual rate r, and time t (in years).

20. In real-world exponential models such as $A = P_0 a^t$, explain the significance of the base a when modelling growth versus decay.

Hard Questions

21. Derive the continuous compound interest formula $A = P_0 e^{rt}$ from the discrete compound interest model and use it to calculate the value of a 1000 dollar investment at an annual rate of 5% after 10 years.

22. A city's population is growing exponentially. Initially it grows at 2% per year, but due to certain policies the rate increases gradually. Describe how you would modify the standard exponential model to account for a growth rate that increases steadily over time.

23. A radioactive substance decays following the model $A = A_0 e^{kt}$. If you are told that after some time the substance is reduced to 20% of its original amount, derive an expression (using logarithms) for the time t in terms of k.

24. Two bank accounts offer the same nominal annual interest rate of 4%. One compounds interest annually and the other monthly. Write the exponential models for both accounts and explain which account will yield a higher return after 20 years.

25. You are given the following data for a city's population: at $t = 0$ years, population is 50 000 and at $t = 8$ years, the population is 65 000. Describe how you would use these data to estimate the exponential growth rate. (Hint: You may plot the data conceptually to assist your explanation.)

26. A radioactive isotope has a half-life of 10 years. Write the exponential decay model in the form $A = A_0 e^{kt}$ and determine the decay constant k. Then, using your model, predict the remaining amount after 25 years.

27. Show that for continuous compounding at rate r, the effective annual interest rate is given by $e^r - 1$. Calculate this effective rate when $r = 0.06$.

28. A bacteria population is decreasing continuously due to environmental factors. If the population halves every 9 hours, determine the continuous decay rate r in the model $N(t) = N_0 e^{rt}$.

29. A scientist models the concentration of a pollutant in a lake using an exponential decay function. If the concentration decreases to 5% of its original value over 15 years, determine the decay constant and calculate approximately how long it will take for the concentration to reduce by 95%.

30. In a short paragraph, discuss the limitations of using exponential models in real-world applications. Suggest at least two modifications or alternative models that might address these limitations.