ILLAWARRA TUTORING

This worksheet introduces the applications of exponential functions. You will learn how to apply the exponential models to real-world scenarios such as compound interest, population growth and radioactive decay. Read each question carefully and answer using the required mathematical procedures.

Easy Questions

1. Write the standard compound interest formula in the form $A = P(1 + r)^n$, and briefly state what each variable represents.

2. A certain bacteria population doubles every fixed period. Write an exponential function of the form $N(t) = N_0\, a^t$ to model a scenario where the population doubles every 10 time units. Specify the value of a.

3. Explain, in your own words, what is meant by the term "half-life" in the context of radioactive decay. Then write an exponential decay function showing that after the half-life period, the original amount is reduced to one-half.

4. Given the formula $A = P\, e^{rt}$ for continuous compounding, list what each of the following variables represents: A, P, r, and t.

5. An investor deposits 1000 dollars into an account that earns 5% annual interest compounded annually. Write down the formula to compute the amount after 3 years and compute the final amount.

Intermediate Questions

6. Joe invests 2000 dollars at an annual interest rate of 3.5% compounded annually. Write the expression for the amount after 10 years, and then calculate the amount.

7. A principal P grows with an interest rate of r per annum compounded semi-annually. Write the corresponding exponential function which expresses the amount after t years.

8. An amount P is invested at an annual interest rate r and is compounded continuously. Write the exponential model using Euler's number, and calculate the amount after 4 years for $P = 1500$ dollars and $r = 0.04$.

9. For a given principal P and annual rate r, explain briefly how the amount accumulated after time t differs between annual compounding $A = P(1+r)^t$ and continuous compounding $A = P\, e^{rt}$.

10. A town's population growing exponentially has an initial population of 5000 and grows at a rate of 2% per annum compounded annually. Write the population function and compute the population after 8 years.

11. A radioactive substance has a half-life of 6 years. Write an expression relating the decay constant λ and the half-life, and then find an expression for λ.

12. An investment account is modelled by the function $A = P\,e^{rt}$. If after 10 years the amount is 3000 dollars and the annual rate is 0.06, set up an equation to determine the initial deposit P and solve it.

13. For an investment with continuous compounding at rate r, derive an expression for the time t required for the investment to triple in value, and then compute t when $r = 0.05$.

14. The exponential decay of a radioactive material is given by $N(t) = N_0\,e^{\lambda t}$. Write an equation in terms of λ that expresses the condition that after time $t_{1/2}$ the remaining amount is half of N_0, and solve for $t_{1/2}$.

15. Carbon-14 decays exponentially. Write the exponential decay model for Carbon-14 given an initial amount N_0 and decay constant λ. Explain briefly how the model is used to date ancient artefacts.

16. A bacteria culture doubles every 4 hours. Write the exponential model for the number of bacteria after t hours, and calculate the number after 12 hours if the initial number is 100.

17. For an investment compounded continuously at rate r, derive the equation to determine the doubling time t. Then, compute the doubling time when $r = 0.07$.

18. A car depreciates in value by a constant percentage every year. Write a general exponential function to model the depreciation of the car's value starting from an initial value V_0.

19. A town's population decreases by 10% per year. Write the exponential decay model for the town's population, and calculate the proportion of the original population remaining after 5 years.

20. A radioactive substance is known to halve its quantity in T years. Write an expression for the decay constant λ in terms of T.

Hard Questions

21. A bank account offers an annual interest rate of 8% compounded quarterly. Write the exponential model, compute the amount factor after one year, and hence determine the effective annual rate.

22. Derive the equation that relates the half-life $t_{1/2}$ of a radioactive substance to its decay constant λ in the formula $A = A_0\,e^{-\lambda t}$. Simplify your answer.

23. A city's population is growing continuously. If the population at time t is given by $P(t) = P_0\, e^{rt}$ and the population increases by 25% over 15 years, find the continuous growth rate r and write the model.

24. An investment of 2000 dollars grows at a nominal annual rate of 5% compounded quarterly. Set up the equation $2000\left(1 + \frac{0.05}{4}\right)^{4t} = 5000$ and solve for t in years.

25. A company's revenue grows continuously at a rate of 4% per annum for the first 5 years and then at 6% per annum for the next 5 years. Write a piecewise exponential function to model the revenue over 10 years.

26. Derive the general formula for the doubling time t_{double} in an exponentially growing account compounded continuously, starting from the formula $A = P\, e^{rt}$.

27. A car worth 30000 dollars depreciates exponentially at a rate of 15% per annum. Write an expression for the value after t years, and determine its value after 3 years.

28. A town's population increases continuously such that over a period of 20 years, the population grows by 50%. Derive the formula to determine the continuous growth rate r, and compute its value.

29. A radioactive substance follows the model $A(t) = A_0\, e^{-\lambda t}$. If after 3 years only 30% of the substance remains:

    (a) Write an equation to determine λ and solve for λ.

    (b) Using the value of λ, predict the percentage remaining after 10 years.

30. A deposit of 5000 dollars earns an annual interest rate of 6%. Compute the amount after 5 years when interest is compounded:

    (a) Continuously, using the model $A = P\, e^{rt}$.

    (b) Quarterly, using the formula $A = P\left(1 + \frac{r}{4}\right)^{4t}$.

    (c) Annually, using the formula $A = P(1 + r)^t$.

    Compare the three amounts and briefly discuss any differences.