ILLAWARRA TUTORING

In this worksheet you will learn to determine the variance and standard deviation of a distribution to measure how spread out the outcomes are. You will practise computing these measures from both raw data and probability distributions, and explore some of their fundamental properties.

Easy Questions

1. Calculate the mean, variance and standard deviation for the dataset 2, 4, 6, 8, assuming that each outcome is equally likely.

2. For the discrete random variable X that takes values 1, 3, and 5 with probabilities 0.2, 0.5, and 0.3 respectively, compute the variance and standard deviation.

3. A random variable X takes the value 7 with probability 1. Find the variance and standard deviation of X.

4. Determine the variance and standard deviation for the set 3, 7 when both outcomes are equally likely.

5. Let X represent a fair coin toss where $X = 0$ indicates tails and $X = 1$ indicates heads. Calculate the variance and standard deviation of X.

Intermediate Questions

6. Find the variance and standard deviation for a discrete random variable X with outcomes -1, 0, and 1 and corresponding probabilities 0.25, 0.5, and 0.25.

7. For the distribution where X takes the values 10, 12, and 14 with probabilities 0.3, 0.4, and 0.3 respectively, calculate the variance and standard deviation.

8. A dataset has the values 4, 8, and 12 with frequencies 3, 5, and 2 respectively. Assuming these frequencies represent relative occurrences, compute the variance and standard deviation.

9. Using the data 1, 2, 3 with equal probability for each outcome, show that the variance equals the mean of the squared deviations from the mean.

10. Let X be a random variable and define $Y = X + c$, where c is a constant. Prove that the variance of Y is the same as that of X; that is, show $\text{Var}(Y) = \text{Var}(X)$.

11. Given a distribution where X takes the values 5, 6, and 7 with probabilities $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{1}{2}$ respectively, compute the variance and standard deviation.
12. Consider a random variable X with outcomes 1 and 2, each with a probability of 0.5. If $Y = 3X$, demonstrate that $\operatorname{Var}(Y) = 9\operatorname{Var}(X)$.
13. For the distribution with X taking values 2, 4, 6, and 8 with probabilities 0.1, 0.2, 0.3, and 0.4 respectively, calculate the variance and standard deviation.
14. Verify that $\operatorname{Var}(X) = E(X^2) - [E(X)]^2$ for the discrete random variable X with outcomes 0, 1, and 2 and probabilities 0.3, 0.4, and 0.3.
15. An exam score dataset consists of scores 50, 60, 70, and 80 with frequencies 1, 2, 2, and 1 respectively. Compute the variance and standard deviation of the scores.
16. For X taking the values -3, -1, 1, and 3, each with probability 0.25, calculate the variance.
17. Determine the variance and standard deviation for a random variable X that takes the values 0, 5, and 10 with probabilities 0.4, 0.4, and 0.2 respectively.
18. If a distribution has a variance of 9, determine its standard deviation and explain your reasoning.
19. Write out the variance formula $\operatorname{Var}(X) = \sum p(x)(x-\mu)^2$ and then use it to compute the variance for the distribution X with outcomes 1, 2, and 3 and probabilities 0.3, 0.4, and 0.3.
20. Given that $\operatorname{Var}(X) = \sum p(x)(x-\mu)^2$, explain why the variance cannot be negative.

Hard Questions

21. Derive the formula $\operatorname{Var}(X) = E\left(X^2\right) - [E(X)]^2$ starting from the definition $\operatorname{Var}(X) = \sum p(x)(x-\mu)^2$.
22. Show, through algebraic manipulation, that for any constant c and random variable X, the variance of cX is $\operatorname{Var}(cX) = c^2 \operatorname{Var}(X)$.
23. For a discrete random variable X with outcomes -2, 0, and 2 occurring with probabilities 0.1, 0.8, and 0.1 respectively, calculate the variance and standard deviation exactly.
24. Prove that if a random variable X has $\operatorname{Var}(X) = 0$, then X is almost surely a constant.
25. Using the definition of variance, explain why it is a measure of spread and compare this with the concept of range for a discrete distribution.
26. Consider the random variable X with outcomes 1, 4, and 9 with probabilities 0.2, 0.5, and 0.3, respectively. Calculate $E(X)$, $E\left(X^2\right)$, and hence find $\operatorname{Var}(X)$ using both $\operatorname{Var}(X) = E\left(X^2\right) - [E(X)]^2$ and the definition $\operatorname{Var}(X) = \sum p(x)\,(x-\mu)^2$.

27. Explain why the standard deviation, defined as the square root of the variance, has the same units as the random variable X, while the variance has squared units. Include a discussion of the formulas.

28. Assume a discrete random variable X has outcomes 2, 4, and 6, each occurring with probability $\frac{1}{3}$. Compute $E(X)$, $E\left(X^2\right)$, $\text{Var}(X)$, and the standard deviation, showing all steps.

29. For a distribution where X takes the values 3, 7, 11, and 15 with probabilities 0.1, 0.2, 0.3, and 0.4, respectively, compute the variance using the shortcut formula $\text{Var}(X) = E\left(X^2\right) - [E(X)]^2$.

30. Prove that the variance satisfies the inequality implied by Jensen's inequality for the convex function $f(x) = x^2$. Use a discrete random variable with two distinct outcomes to illustrate your proof.