ILLAWARRA TUTORING

In this worksheet, students will learn to define discrete random variables and understand how their distributions work. You will explore how sample space outcomes are mapped to numerical values and how a probability mass function summarises the distribution of a discrete random variable. Answer each question carefully and show your working where necessary.

Easy Questions

1. Define a discrete random variable. Write your answer in your own words.
2. Provide one example of a discrete random variable from everyday life and briefly explain why it is discrete.
3. Consider the experiment of tossing a coin once with the sample space H, T. Define a random variable X that assigns 1 to a head and 0 to a tail. List the possible values of X.
4. When rolling a standard die, let X be the random variable representing the number shown. List all possible outcomes for X.
5. Explain why the number of siblings a person has is a discrete random variable.

Intermediate Questions

6. Consider an experiment where a bag contains three coloured balls: red, blue, and green. A ball is drawn at random and then replaced. Define a random variable Y that takes a numerical value based on the colour (red=1, blue=2, green=3). List the possible outcomes of Y.
7. A random variable X takes values 1, 2, and 3 with probabilities $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{5}{12}$ respectively. Check if these probabilities define a valid probability mass function.
8. State the two essential conditions that a function $p(x)$ must satisfy in order to be a valid probability mass function for a discrete random variable.
9. Let X be a random variable that takes the values 1, 2, and 3, with $p(1) = \frac{1}{4}$, $p(2) = \frac{1}{2}$, and $p(3) = \frac{1}{4}$. Verify that this distribution is valid.
10. Consider tossing a coin three times. Define the random variable X as the number of heads observed. List all possible values that X can take.

11. In a quality control experiment, let X be the number of defective items found in a small sample from a production line. Describe the sample space for X if the maximum number of defects possible is 3.

12. Explain how a random variable is a function from a sample space to the real numbers. Provide a simple example to illustrate your answer.

13. An experiment has outcomes A, B, C, D with equal probability. Define a random variable Y that assigns $Y(A) = 1$, $Y(B) = 2$, $Y(C) = 3$, and $Y(D) = 4$. Write down the probability mass function for Y.

14. In a short paragraph, explain the difference between an outcome of an experiment and a random variable.

15. Write the formal definition of a discrete random variable as a function from a sample space S to a set of numbers, including the role of the probability mass function.

16. Decide whether the following is a discrete random variable: the time, in minutes, a student takes to complete a test. Explain your answer.

17. Suppose you roll two standard dice and define a random variable X as the sum of the numbers on the two dice. List the range of possible values for X.

18. Explain in a few sentences why the probabilities in a probability mass function must sum to 1.

19. Describe how the probability mass function assigns probabilities to the outcomes of a discrete random variable.

20. A discrete random variable X takes the values 0, 1, and 2. If $p(0) = 0.5$ and $p(1) = 0.3$, determine $p(2)$.

Hard Questions

21. Prove that if X is a discrete random variable with probability mass function $p(x)$, then for any event $A \subseteq S$,

$$P(X \in A) = \sum_{x \in A} p(x).$$

Explain each step of your reasoning.

22. A ball numbered n is drawn from an urn containing numbers 1 to 10 with equal likelihood. A random variable X is defined such that

$$X = \begin{cases} n, & \text{if } n \text{ is even} \\ n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Write down the probability mass function for X.

23. For the random variable X defined in Question 22, list all the possible values that X can take and determine the corresponding probabilities.

24. A random variable Y is defined by the following table:

y	$p(y)$
0	0.1
2	0.3
3	0.4
5	0.2

Explain how you would verify that this table represents a valid probability mass function and state the range of Y.

25. Suppose a discrete random variable Z is defined on a finite sample space S with equally likely outcomes. Write a general expression for $p(z)$ in terms of the number of outcomes corresponding to $Z = z$ and the total number of outcomes in S.

26. Let A be an event defined in terms of a discrete random variable X. Explain why the probability $P(X \in A)$ is computed by summing the values $p(x)$ for all $x \in A$.

27. In a short paragraph, explain in your own words the role of the probability mass function in summarising the behaviour of a discrete random variable.

28. Consider an experiment where a number from 1 to n is selected at random with each number being equally likely. Define a random variable X that outputs 1 if the selected number is prime and 0 otherwise. Express the probability mass function of X in terms of n and the number of primes within $[1, n]$.

29. Discuss what issues might arise if the sum of the probabilities in a probability mass function does not equal 1. Support your discussion with concepts from probability theory.

30. A spinner with 8 equal sectors numbered 1 to 8 is spun. A random variable T is defined such that if the spinner lands on an odd number, then T is one less than the obtained number; if it lands on an even number, then T equals the number displayed. Write the probability mass function for T and list all possible values of T.