ILLAWARRA TUTORING

In this worksheet you will explore the binomial distribution. You will learn how to use the binomial probability formula in situations involving a fixed number of independent trials and two possible outcomes. Work through the questions, showing all your working.

Easy Questions

1. Define a binomial experiment. In your answer, list the necessary conditions that distinguish a binomial experiment. (Hint: Consider the number of trials, independence, constant probability, and only two outcomes per trial: success or failure.)
2. State whether the following experiment is a binomial experiment: A fair coin is flipped five times, and the number of heads is recorded. Explain your answer in one or two sentences.
3. Write the formula for the probability of exactly k successes in n independent trials, each with success probability p. Include a brief explanation of the combination notation that appears in the formula.
4. Calculate the probability of obtaining exactly 3 successes in 5 independent trials when the probability of success on each trial is 0.5. Show your working.
5. Use the binomial formula to compute the probability of obtaining no successes in 8 trials when the probability of success is 0.2.

Intermediate Questions

6. Calculate the probability of obtaining exactly 4 successes in 7 trials when $p = 0.3$. Include all steps clearly.
7. For a binomial experiment with $n = 15$ trials and success probability 0.4, find the probability of obtaining exactly 10 successes.
8. Determine the probability of achieving exactly 5 successes in 10 trials when $p = 0.6$. Show your working.
9. A fair coin (with $p = 0.5$ for heads) is tossed 8 times. Compute the probability of obtaining exactly 6 heads.
10. Explain what is meant by a Bernoulli trial and discuss how a sequence of Bernoulli trials forms the basis of the binomial distribution.

11. In a binomial experiment with $n = 5$ and $p = 0.3$, calculate the cumulative probability of obtaining at most 2 successes. (That is, find the probability for $k = 0, 1, 2$ and sum them.)

12. For $n = 4$ trials with $p = 0.25$, show by calculation that the probability of obtaining at least one success is 1 minus the probability of no successes.

13. In an experiment with $n = 6$ and $p = 0.5$, use the complement rule to calculate the probability of obtaining at least 3 successes. (Hint: Compute the probability of fewer than 3 successes and subtract from 1.)

14. For $n = 9$ and $p = 0.7$, determine the probability of attaining exactly 9 successes. Explain your reasoning.

15. For $n = 12$ trials with $p = 0.2$, compute the probability of obtaining exactly 0 successes.

16. For a binomial distribution with $n = 3$ and $p = 0.5$, calculate the probability for each possible number of successes and verify that their sum equals 1.

17. Although the expected value formula was covered earlier, use the binomial model to determine the expected number of successes in 10 trials with $p = 0.4$. Provide a brief explanation of your answer.

18. In a manufacturing process, the probability that an item is defective is 0.05. In a sample of 20 items, calculate the probability that exactly 1 item is defective.

19. A multiple-choice question offers 4 choices, with only one correct answer. If a student guesses randomly, the probability of a correct answer is 0.25. Find the probability that, in answering 8 independent questions, the student gets exactly 2 correct.

20. A basketball player makes a free throw with probability 0.8. Calculate the probability that out of 15 free throw attempts he makes exactly 12. Clearly show your calculations.

Hard Questions

21. Prove that the binomial probability distribution is valid by showing that $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$ for any integer $n \geq 1$ and $0 \leq p \leq 1$. Provide a clear explanation.

22. Derive the general binomial probability formula from first principles by explaining the role of combinations and the probabilities of individual successes and failures.

23. For an experiment with $n = 10$ and $p = 0.3$, compute the probability of obtaining an even number of successes. That is, sum the probabilities for $k = 0, 2, 4, 6, 8, 10$. (You may leave your answer in summation notation if needed.)

24. Prove the combinatorial identity $\binom{n}{k} = \binom{n}{n-k}$ and explain briefly how this property is useful when working with binomial distributions.

25. In a binomial experiment with $n = 12$ trials and success probability p, suppose that the probability of exactly 2 successes equals the probability of exactly 10 successes. Determine the value of p. Show all algebraic steps.

26. For a binomial distribution with $n = 7$ trials and unknown p, if the probability of exactly 3 successes is maximised, find the value of p that maximises this probability. Explain your reasoning.

27. Explain why the binomial distribution approaches a normal distribution as the number of trials increases. In your explanation, refer to the conditions that facilitate a normal approximation. (Do not perform any numerical approximations; focus on the conceptual aspects.)

28. A biased coin with probability of heads p is tossed 5 times. If the probability of obtaining exactly 3 heads is equal to the probability of obtaining exactly 2 heads, determine p. Provide a clear solution.

29. Let n be an integer greater than 1. Using the binomial theorem, show that the expected number of successes in n trials is np. Provide a detailed derivation.

30. Derive an expression for the variance of the binomial distribution $\sigma^2 = np(1 - p)$ using summation notation. Your answer should detail each step of your derivation.