ILLAWARRA TUTORING

This worksheet explores the binomial distribution. Students will investigate situations with a fixed number of independent trials, each with two possible outcomes. They will learn to calculate probabilities, the mean and variance of a binomial experiment, and examine properties and applications of the binomial model.

Easy Questions

1. List the four conditions that must be satisfied for a random experiment to be modelled by a binomial distribution. Use n for the number of trials, p for the probability of success, and state the requirement for independence and constant probability.

2. Consider a binomial experiment with $n = 5$ trials and probability of success $p = 0.3$. Calculate the probability of obtaining exactly $k = 2$ successes using the binomial probability formula $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$.

3. Provide one example of a real-world situation that can be modelled by a binomial distribution. Clearly state the number of trials, the definition of success, and the probability of success in your example.

4. For a binomial experiment with $n = 10$ and $p = 0.4$, use the formula for the expected value $\mu = n \cdot p$ to calculate the mean number of successes.

5. For the same experiment as in Question 4, compute the variance using the formula $\sigma^2 = n \cdot p \cdot (1-p)$.

Intermediate Questions

6. Explain how the binomial probability formula $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ is derived. In your explanation, describe the significance of the binomial coefficient.

7. Calculate $\binom{8}{3}$ and explain what this number represents in a binomial experiment with $n = 8$ trials.

8. A quality-control inspector checks 12 light bulbs, and each has a 0.1 probability of being defective, independent of the others. Explain why or why not this situation can be modelled with a binomial distribution.

9. For a situation with $n = 7$ trials and $p = 0.2$, calculate the probability of obtaining at least one success by using the complement rule.

10. For a binomial experiment with $n = 6$ and $p = 0.5$, calculate the probability of obtaining at most 2 successes. Show your calculations.

11. Consider $n = 8$ and $p = 0.3$. Compute the cumulative probability of obtaining at most 3 successes, i.e. $P(X \leq 3)$.

12. A fair coin is tossed 4 times. Write down the probability of getting exactly 2 heads using the binomial formula. Explain why this is a binomial experiment.

13. In a production process, the probability of producing a defective item is 0.05. If 20 items are produced, calculate the probability that exactly 1 item is defective.

14. For a binomial distribution with $n = 10$ and $p = 0.5$, compute the probability of obtaining exactly 3 successes.

15. Write a brief explanation of why independence of trials is important in the binomial distribution and state one scenario where this assumption might be violated.

16. Suppose in a binomial experiment with $n = 8$, the probability of getting exactly 3 successes is 0.25. Using the formula $\binom{8}{3} p^3 (1-p)^5 = 0.25$, explain the steps you would take to solve for p, even if the algebra is intricate.

17. Consider two binomial experiments: Experiment A with $n = 10$, $p = 0.3$ and Experiment B with $n = 20$, $p = 0.3$. Compare how the probability of exactly 3 successes differs between the two experiments by discussing the effect of increasing n.

18. For an experiment with $n = 15$ and $p = 0.2$, calculate the probability of obtaining less than or equal to 2 successes. Explain the steps involved.

19. On a blank sheet of paper, sketch the probability distribution (bar graph) for a binomial experiment with $n = 5$ and $p = 0.5$. Label the horizontal axis with the number of successes and the vertical axis with the corresponding probabilities.

20. For a fixed number of trials n, discuss what happens to the variance $\sigma^2 = n \cdot p \cdot (1-p)$ if the value of p changes from 0.2 to 0.8. Explain your reasoning.

Hard Questions

21. Prove algebraically that the sum of the probabilities for a binomial distribution over all possible outcomes is 1, i.e. show that $\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$.

22. For $n = 12$ and $p = 0.4$, find the probability that the number of successes lies between 4 and 7 inclusive. Provide a detailed solution.

23. A binomial experiment has $n = 18$ trials and an unknown p. If it is given that the mean is 6.3 and the variance is 3.969, determine the value of p. Show all steps.

24. Show that for a binomial experiment with $p = 0.5$, the probability of k successes is equal to the probability of $n - k$ successes. Include an explanation using the binomial formula.

25. Given $n = 14$ and $p = 0.5$, demonstrate that $P(X = k) = P(X = 14 - k)$ for any valid k. Provide an algebraic justification.

26. A diagnostic test correctly identifies a disease with probability 0.92. If the test is administered to 15 patients who all have the disease, calculate the probability that at least 13 patients are correctly diagnosed. Use the binomial distribution in your solution.

27. In a game, a player must score at least 4 successes out of 8 independent trials (each with $p = 0.35$) to win. Find the probability that the player wins. Explain your working.

28. For $n = 10$ and $p = 0.6$, calculate the probability that there are between 5 and 7 successes (inclusive). Explain how you combined individual probabilities.

29. For a binomial distribution with parameters n and p, the skewness is given by $\gamma = \dfrac{1-2p}{\sqrt{n \cdot p \cdot (1-p)}}$. For $n = 25$ and $p = 0.3$, calculate the skewness and interpret its meaning.

30. Derive a recursion relation for the binomial probabilities, showing how $P(X = k+1)$ can be expressed in terms of $P(X = k)$. Begin with the formula $P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ and simplify your result.